Computable Real Arithmetic Using Linear Fractional Transformations

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Abstract

This report describes my progress to date and plans for the future in the search for a feasible representation of computable real numbers in terms of linear fractional transformations and a set of algorithms for the elementary functions: rational, power, trigonometric, inverse trigonometric, exponential and logarithmic, hyperbolic and inverse hyperbolic.

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1 Introduction

A feasible representation of the computable real numbers must be simple and yet algorithmically rich. Linear fractional transformations (lfts) provide an ideal setting because it is intimately related to the idea of a computable real numbers and is closely related to continued fractions, which provide a plethora of algorithms for the elementary functions.

In section 2, we highlight the axioms for the real numbers and define the notion of a computable real number.

In section 3, we show how the computable real numbers can be represented incrementally by sequences of 2×2 matrices or equivalently by sequences of linear fractional transformations.

In section 4, we digress slightly in order to examine the notion of a Padé approximant, which provide a more comprehensive method of approximating analytic functions than Taylor expansions.

In section 5, we examine the remarkable notion of a continued fraction and show how they can be transformed into sequences of linear fractional transformations. We also look at three ways to represent analytic functions using continued fractions and illustrate the relationship between these and Padé approximants.

In section 6, we delve into the world of the hypergeometric function, which allows all the elementary functions to be encapsulated by a single all encompassing function. Hopefully, this may provide a means to formulate a single algorithm to encompass all the elementary functions and many others as well. It is satisfying at this stage to see an intimate relationship between the hypergeometric function and linear fractional transformations.

In section 7, we tentatively put forward an embryonic operation semantics involving potentially arbitrary dimension tensor arithmetic.

In section 8, we begin to address the potential problem of exploding integers by using B-iadic numbers.

In section 9, we evaluate numerous formulae for the elementary functions in terms of infinite sequences of tensors.

2 The Properties of Real Numbers

2.1 Axioms for the Real Numbers

It seems incomplete to talk about computable real numbers without first laying out the definition for a real number. The real numbers $\mathbb{R}$ are completely characterised by the three groups of axioms below.

The Field Axioms: For all real numbers x, y, z

$$\begin{aligned} x+y &= y+x && (1)\\ (x+y)+z &= x+(y+z) && (2)\\ x+0 &= x && (3)\\ x+(-x) &= 0 && (4)\\ x\times y &= y\times x && (5)\\ (x\times y)\times z &= x\times(y\times z) && (6)\\ x\times 1 &= x && (7)\\ x\times(x^{-1}) &= 1 && (8)\\ x\times(y+z) &= (x\times y)+(x\times z) && (9) \end{aligned}$$

Any set that satisfies these axioms is called a field (under $+$ and $\times$).

Axioms of Order: The subset P of positive real numbers satisfies

$$\begin{aligned} x,y\in P &\Rightarrow x+y\in P && (10)\\ x,y\in P &\Rightarrow x\times y\in P && (11)\\ x\in P &\Rightarrow -x\notin P && (12)\\ x\in P &\Rightarrow x=0 \text{ or } x\in P \text{ or } -x\in P && (13) \end{aligned}$$

Any set that satisfies these axioms as well is called an ordered field. We define the notion $x<y$ to mean $y+(-x)\in P$.

Completeness Axiom: Every nonempty set S of real numbers which has an upper bound has a least upper bound. It is this axiom that distinguishes the real numbers from the rational numbers.

2.2 The Extended Real Numbers

It is often convenient to extend the system of real numbers by the addition of the three elements, $\pm\infty$ and $\perp$. This enlarged set is called the set of *extended real numbers*. We extend the definition of $<$ to the extended real numbers by stipulating $-\infty<x<\infty$, for each real number x. We also define

$$\begin{aligned} x+\infty &= \infty && (14)\\ x+(-\infty) &= -\infty && (15)\\ x\times\infty &= \infty \text{ if } x>0 && (16)\\ x\times(-\infty) &= -\infty \text{ if } x>0 && (17) \end{aligned}$$

for all real numbers x, and set

$$\begin{aligned}
\infty + \infty &= \infty && (18)\\
(-\infty) + (-\infty) &= -\infty && (19)\\
\infty \times (\pm\infty) &= (\pm\infty) && (20)\\
(-\infty) \times (\pm\infty) &= (\mp\infty) && (21)\\
\infty + (-\infty) &= \bot && (22)\\
0 \times (\pm\infty) &= 0. && (23)
\end{aligned}$$

2.3 The Computable Real Numbers

The set of all computer programs is countable. However, the real numbers are uncountable. Therefore, we cannot hope to represent all real numbers on a computer. We must consider a countable subset of the real numbers, namely the computable real numbers, which is closed under the basic arithmetic operations and the elementary functions.

Definition 2.1 *A real number x is* computable *if and only if there exists a recursively enumerable sequence $\left\langle \left[\frac{a_n}{c_n}, \frac{b_n}{d_n}\right] \right\rangle_{n=0}^{\infty}$ of nested rational intervals*

$$\left[\frac{a_0}{b_0}, \frac{c_0}{d_0}\right] \supseteq \left[\frac{a_1}{b_1}, \frac{c_1}{d_1}\right] \supseteq \left[\frac{a_2}{b_2}, \frac{c_2}{d_2}\right] \supseteq \cdots$$

enclosing x such that the sequence of the interval lengths $\left\langle \left| \frac{a_n}{b_n} - \frac{c_n}{d_n} \right| \right\rangle_{n=0}^{\infty}$ converges to 0 where $a_n, b_n, c_n, d_n \in \mathbb{Z}$.

Allowing c_n and d_n to be zero means that we can represent unbounded intervals. It follows that

$$x = \bigcap_{n \geq 0} \left[\frac{a_n}{b_n}, \frac{c_n}{d_n}\right].$$

Allow infinite bounds.

2.4 B-adic Numbers

Definition 2.2 *If B is an integer greater than or equal to 2, a real number r is called a B-*adic number *if there exists two integers p and q such that $r = \frac{p}{B^q}$.*

3 Representation of Real Numbers

3.1 Matrix Representation

An incremental representation of the computable real numbers would be ideal, since this naively implies less computation.

Define the *embedding* function $E : I \to \mathbb{M}$ by

$$E\left(\left[\frac{a}{b}, \frac{c}{d}\right]\right) := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{24}$$

where the rational numbers are in their lowest form.

Define the *projection* function $P : \mathbb{M} \to I$ by

$$P\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) := \left[\min\left(\frac{a}{b}, \frac{c}{d}\right), \max\left(\frac{a}{b}, \frac{c}{d}\right)\right] \tag{25}$$

This leads naturally to an *equivalence* relation $\equiv$ on matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv \begin{pmatrix} e & g \\ f & h \end{pmatrix} \iff P\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = P\left(\begin{pmatrix} e & g \\ f & h \end{pmatrix}\right) \tag{26}$$

and an *ordering* relation $\sqsubseteq$ on matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \sqsubseteq \begin{pmatrix} e & g \\ f & h \end{pmatrix} \iff P\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) \supseteq P\left(\begin{pmatrix} e & g \\ f & h \end{pmatrix}\right) \tag{27}$$

and the notion of meet and join

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \wedge \begin{pmatrix} e & g \\ f & h \end{pmatrix} := E\left([\Phi(\min, \min), \Phi(\max, \max)]\right) \tag{28}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \vee \begin{pmatrix} e & g \\ f & h \end{pmatrix} := \tag{29}$$

$$\begin{cases} E\left([\Phi(\max, \min), \Phi(\min, \max)]\right) & \text{if } \Phi(\max, \min) <= \Phi(\min, \max) \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases} \tag{30}$$

$$\text{where} \quad \Phi(f, g) = f\left(g\left(\frac{a}{b}, \frac{c}{d}\right), g\left(\frac{e}{f}, \frac{g}{h}\right)\right). \tag{31}$$

Therefore, a sequence of rational intervals $\left\langle \left[\frac{a_n}{c_n}, \frac{b_n}{d_n}\right] \right\rangle_{n=0}^{\infty}$ can be seen as a chain of matrices

$$\begin{pmatrix} a_0 & c_0 \\ b_0 & d_0 \end{pmatrix} \sqsubseteq \begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} \sqsubseteq \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix} \sqsubseteq \cdots.$$

We need only consider the unique rational intervals of a sequence representing a computable real number. This subsequence, which may be finite or infinite, will be denoted by $\left\langle \left[\frac{a_n}{b_n}, \frac{c_n}{d_n}\right] \right\rangle_{n=0}^{m}$ where $m \in \mathbb{N} \cup \{\infty\}$. In this way, we need only consider non-singular matrices, except possible the last one, which we can replace by a vector anyway.

Therefore, we can devise another sequence of matrices $\left\langle \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{pmatrix} \right\rangle_{n=0}^{m}$ to allow us to incrementally determine the sequence of unique rational intervals.

$$\begin{pmatrix} \alpha_0 & \gamma_0 \\ \beta_0 & \delta_0 \end{pmatrix} := \begin{pmatrix} a_0 & c_0 \\ b_0 & d_0 \end{pmatrix} \tag{32}$$

$$\begin{pmatrix} \alpha_{n+1} & \gamma_{n+1} \\ \beta_{n+1} & \delta_{n+1} \end{pmatrix} := k_n \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix}^{-1} \begin{pmatrix} a_{n+1} & c_{n+1} \\ b_{n+1} & d_{n+1} \end{pmatrix}. \tag{33}$$

Without loss of generality, we shall assume that all c_n and d_n have the same sign.

Clearly, this matrix representation is highly redundant on account of the scaling invariance of fractions, manifested by the arbitrary constants k_n. This means that we can easily choose integer values for α_n, β_n, γ_n and δ_n, by setting

$$k_n := -\begin{vmatrix} a_n & c_n \\ b_n & d_n \end{vmatrix} = b_n c_n - a_n d_n \tag{34}$$

which we note is always positive.

Definition 3.1 *An* infinite normal product *is a infinite sequence of matrices* $\left\langle \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{pmatrix} \right\rangle_{n=0}^{\infty}$ *satisfying*

$$\begin{aligned} \alpha_0, \beta_0, \gamma_0, \delta_0 &\in \mathbb{Z} \\ \beta_0 \times \delta_0 &\geq 0 \\ \alpha_{n>0}, \beta_{n>0}, \gamma_{n>0}, \delta_{n>0} &\in \mathbb{N} \\ \begin{vmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{vmatrix} &\neq 0 \end{aligned} \tag{35}$$

Definition 3.2 *A* finite normal product *is a finite sequence of matrices* $\left\langle \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{pmatrix} \right\rangle_{n=0}^{m}$ *terminated by a vector* $\begin{pmatrix} \alpha_{m+1} \\ \beta_{m+1} \end{pmatrix}$ *satisfying*

$$\begin{aligned} \alpha_0, \beta_0, \gamma_0, \delta_0 &\in \mathbb{Z} \\ \beta_0 \times \delta_0 &\geq 0 \\ \alpha_{n>0}, \beta_{n>0}, \gamma_{n>0}, \delta_{n>0} &\in \mathbb{N} \\ \begin{vmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{vmatrix} &\neq 0 \end{aligned} \tag{36}$$

Proposition 3.3 *The matrix representation* $\left\langle \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{pmatrix} \right\rangle_{n=0}^{m}$ *for a sequence of strictly nested rational intervals* $\left\langle \left[\frac{a_n}{b_n}, \frac{c_n}{d_n}\right] \right\rangle_{n=0}^{m+1}$ *corresponds to a normal product satisfying*

$$\prod_{n=0}^{r} \begin{pmatrix} \alpha_n & \gamma_n \\ \beta_n & \delta_n \end{pmatrix} \equiv \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix}. \tag{37}$$

for $0 \leq r \leq m$.

Proof From (33) and (34), we have

$$\begin{pmatrix} \alpha_{n+1} & \gamma_{n+1} \\ \beta_{n+1} & \delta_{n+1} \end{pmatrix} = \begin{pmatrix} -d_n & c_n \\ b_n & -a_n \end{pmatrix} \begin{pmatrix} a_{n+1} & c_{n+1} \\ b_{n+1} & d_{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} c_n b_{n+1} - d_n a_{n+1} & c_n d_{n+1} - d_n c_{n+1} \\ b_n a_{n+1} - a_n b_{n+1} & b_n c_{n+1} - a_n d_{n+1} \end{pmatrix}$$

But, we know that

$$\frac{a_n}{b_n} \leq \frac{a_{n+1}}{b_{n+1}} < \frac{c_{n+1}}{d_{n+1}} \leq \frac{c_n}{d_n}.$$

Therefore, we have

$$\alpha_{n\geq 0}, \beta_{n\geq 0}, \gamma_{n\geq 0}, \delta_{n\geq 0} \in \mathbb{N}.$$

The rest of the proof is trivial. □

Proposition 3.4 *A normal product represents a sequence of strictly nested intervals.*

Proof It is sufficient to consider

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

where $a, b \in \mathbb{Z}$ and $c, d, e, f \in \mathbb{N}$ and

$$\frac{a}{b} < \frac{c}{d}$$

and show rather trivially that

$$\frac{a}{b} < \frac{ae + cf}{be + df} < \frac{c}{d}$$

and $ae + cf \in \mathbb{Z}$ and $be + df \in \mathbb{N}$. □

3.2 Linear Fractional Transformations

Definition 3.5 *An* N-dimensional lft $t_A(x_1, x_2, \ldots, x_N)$ *has the form*

$$\begin{aligned} t_A : \mathbb{R}^N &\rightarrow \mathbb{R} \\ t_A(x_1, x_2, \ldots, x_N) &:= \frac{\sum_{i_1, i_2, \ldots, i_N \in \{0,1\}} A_{i_1 i_2 \cdots i_N 0} \prod_{j=1}^{N} x_j^{1-i_j}}{\sum_{i_1, i_2, \ldots, i_N \in \{0,1\}} A_{i_1 i_2 \cdots i_N 1} \prod_{j=1}^{N} x_j^{1-i_j}} \end{aligned} \tag{38}$$

where $A_{i_0 i_1 i_2 \cdots i_N} \in \mathbb{Z}$ are the coefficients of a rank $(N+1)$ tensor A.

Due to the 2-dimensional limitation of the human retina, we shall denote a rank $(N+1)$ tensor A by

$$\begin{pmatrix} A_{00\cdots 00} & A_{00\cdots 10} & \cdots & A_{11\cdots 10} \\ A_{00\cdots 01} & A_{00\cdots 11} & \cdots & A_{11\cdots 11} \end{pmatrix}$$

Remark 3.6 *A 0-dimensional lft is a rational number*

$$t\begin{pmatrix} a \\ b \end{pmatrix} := \frac{a}{b}. \tag{39}$$

A 1-dimensional lft has the general form

$$t\begin{pmatrix} a & c \\ b & d \end{pmatrix}(x) := \frac{ax+c}{bx+d} \tag{40}$$

and a 2-dimensional lft has the general form

$$t\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}(x,y) := \frac{axy+cx+ey+g}{bxy+dx+fy+h}. \tag{41}$$

Definition 3.7 *Let the r-composition $\circ_r$ of an N-dimensional lft t_A and an M-dimensional lft t_B be given by*

$$\begin{aligned} & t_A \circ_r t_B(x_1, x_2, \ldots, x_{(N+M-1)}) \\ = \;& t_A(x_1, x_2, \ldots, x_{r-1}, \\ & t_B(x_r, x_{(r+1)}, \ldots, x_{(r+M-1)}), \\ & x_{(r+M)}, \ldots, x_{(N+M-1)}) \end{aligned} \tag{42}$$

Definition 3.8 *Let the r-product $\bullet_r$ of a rank $(N+1)$ tensor A and a rank $(M+1)$ tensor B be given by*

$$(A \bullet_r B)_{i_1 \cdots i_{(N+M)}} = \sum_{k \in \{0,1\}} A_{i_1 \cdots i_{(r-1)} k i_{(r+M)} \cdots i_{(N+M-1)}} B_{i_r \cdots i_{(r+M-1)} k} \tag{43}$$

Proposition 3.9 *The r-composition of lfts is equivalent to the r-products of tensors.*

$$t_A \circ_r t_B = t_{(A \bullet_r B)} \tag{44}$$

Corollary 3.10 *The composition of 1-dimensional lfts is equivalent to 2×2 matrix multiplication.*

Proof

$$\begin{aligned} t\begin{pmatrix} a & c \\ b & d \end{pmatrix} \circ t\begin{pmatrix} e & g \\ f & h \end{pmatrix}(x) &= \frac{a\left(\frac{ex+g}{fx+h}\right)+c}{b\left(\frac{ex+g}{fx+h}\right)+d} \\ &= \frac{(ae+cf)x+(ag+ch)}{(be+df)x+(bg+dh)} \\ &= t\begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}(x) \end{aligned}$$

Compare this with matrix multiplication

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bullet \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}$$

□

Therefore, we can freely move between lfts and matrices.

4 Padé Approximants

Definition 4.1 *We denote the L, M* Padé Approximant *[4] to $f(x)$ by*

$$[L/M] := \frac{P_L(x)}{Q_M(x)} \tag{45}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{46}$$

determines the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation

$$f(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}). \tag{47}$$

Since we can obviously multiply the numerator and denominator by any constant and leave $[L/M]$ unchanged, we impose the normalisation condition $Q_M(0) = 1$. Finally we require that $P_L(x)$ and $Q_M(x)$ have no common factors.

Theorem 4.2 (uniqueness) *When it exists, the $[L/M]$ Padé approximant to any formal power series $f(x)$ is unique.*

4.1 The Padé Table

Padé [8] emphasised the importance of displaying the Padé approximants in tabular form. By the Padé table we mean the array

$$\begin{array}{cccccc}
[0/0] & [0/1] & [0/2] & [0/3] & [0/4] & \cdots \\
[1/0] & [1/1] & [1/2] & [1/3] & [1/4] & \cdots \\
[2/0] & [2/1] & [2/2] & [2/3] & [2/4] & \cdots \\
[3/0] & [3/1] & [3/2] & [3/3] & [3/4] & \cdots \\
[4/0] & [4/1] & [4/2] & [4/3] & [4/4] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

4.2 The C Table

By the C table to the formal power series (46), we mean the array

$$\begin{array}{cccccc}
C(0/0) & C(0/1) & C(0/2) & C(0/3) & C(0/4) & \cdots \\
C(1/0) & C(1/1) & C(1/2) & C(1/3) & C(1/4) & \cdots \\
C(2/0) & C(2/1) & C(2/2) & C(2/3) & C(2/4) & \cdots \\
C(3/0) & C(3/1) & C(3/2) & C(3/3) & C(3/4) & \cdots \\
C(4/0) & C(4/1) & C(4/2) & C(4/3) & C(4/4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

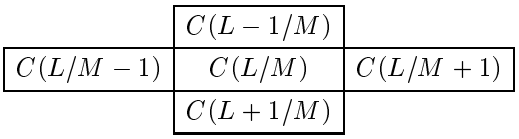

Figure 1: The relative position of the elements in the C table equated by identity (49).

where we define [with Frobenius [3]]

$$C(r/s) = \det \begin{vmatrix} a_{r-s+1} & a_{r-s+2} & \cdots & a_r \\ a_{r-s+2} & a_{r-s+3} & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_r & a_{r+1} & \cdots & a_{r+s-1} \end{vmatrix} . \tag{48}$$

The identity [4]

$$C(L/M)^2 = C(L+1)C(L-1/M) - C(L/M+1)C(L/M-1) \tag{49}$$

together with

$$C[L/0] = 1 \tag{50}$$

$$C[L/1] = a_L \tag{51}$$

$$C[0/M] = (-1)^{\frac{1}{2}M(M-1)} a_0^M \tag{52}$$

provide an efficient way to compute the elements in a C table [see Fig. (1)].

5 Continued Fractions

The development

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \ddots}}}}} \tag{53}$$

is called a *continued fraction.*

The quantity

$$\frac{P_n}{Q_n} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots + \frac{b_{n-1}}{a_n}}}} \tag{54}$$

is called the nth *approximant.* The 0th approximant is a_0.

5.1 Continued Fractions as the Composition of Linear Fractional Transformations

Using the lfts

$$t_{A_n}(x) := a_n + \frac{b_n}{x} \tag{55}$$

where

$$A_n := \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}, \tag{56}$$

we can generate the continued fraction

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}}. \tag{57}$$

Observe that the nth *approximant* is given by

$$\frac{P_n}{Q_n} = t_{A_0} \circ t_{A_1} \circ \cdots \circ t_{A_n}(\infty) = t_{A_0} \circ t_{A_1} \circ \cdots \circ t_{A_{n+1}}(0) \tag{58}$$

and that the value of the continued fraction is

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = \lim_{n\to\infty} t_{A_0} \circ t_{A_1} \circ \cdots \circ t_{A_n}(\infty) = \lim_{n\to\infty} t_{A_0} \circ t_{A_1} \circ \cdots \circ t_{A_n}(0) \tag{59}$$

provided that it is convergent.

We shall refer to continued fractions with positive coefficients by which we will mean, slightly misleadingly, that $a_0, b_0 \in \mathbb{Z}$ and $a_{n>0}, b_{n>0} \in \mathbb{N}$. This subset of the continued fractions has the extremely useful property that the sequence of approximants oscillate back and forth.

$$\cdots \leq \frac{P_n}{Q_n} \leq \frac{P_{n+2}}{Q_{n+2}} \leq \frac{P_{n+4}}{Q_{n+4}} \leq \cdots \leq \frac{P_{n+5}}{Q_{n+5}} \leq \frac{P_{n+3}}{Q_{n+3}} \leq \frac{P_{n+1}}{Q_{n+1}} \leq \cdots \tag{60}$$

This is obviously extremely useful from the point of view of computable real numbers. If we consider the lfts (55) in matrix notation, we can further observe that

$$\prod_{n=0}^{m} \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} P_m & P_{m-1} \\ Q_m & Q_{m-1} \end{pmatrix} \tag{61}$$

where $Q_{-1} = 0$, and hence

$$\prod_{n=0}^{m} \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix} \sqsubseteq \prod_{n=0}^{m+1} \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}. \tag{62}$$

Therefore, a continued fraction with positive coefficients in matrix notation corresponds to a normal product.

Of course, we need to consider the issue of convergence. The following comprehensive test was devised by Pringsheim [10].

Lemma 5.1 *The divergence of the series*

$$\sum_{n=2}^{\infty} \sqrt{\frac{b_{n-1} b_n}{a_n}} \tag{63}$$

is necessary and sufficient for the convergence of the continued fraction (53) with positive coefficients.

5.2 Equivalence Transformations

Unfortunately, by only considering the subset of continued fractions with positive coefficients, we exclude many useful algorithms for some of the elementary functions. The so-called *equivalence transformation*

$$a_0 + \cfrac{c_0 b_0}{c_0 a_1 + \cfrac{c_0 c_1 b_1}{c_1 a_2 + \cfrac{c_1 c_2 b_2}{c_2 a_3 + \cfrac{c_2 c_3 b_3}{c_3 a_4 + \cfrac{c_3 c_4 b_4}{c_4 a_5 + \ddots}}}}} \tag{64}$$

allows us to transform some continued fractions to the positive coefficients form. In matrix notation, we have

$$\prod_{n=0}^{m} \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} a_0 & c_0 b_0 \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{m} \begin{pmatrix} c_{n-1} a_n & c_{n-1} c_n b_n \\ 1 & 0 \end{pmatrix} \tag{65}$$

for all $m \in \mathbb{N}$.

In fact, it is easy to see that this follows from the simple matrix transformation

$$\begin{pmatrix} a & c \\ b & 0 \end{pmatrix} \begin{pmatrix} d & f \\ e & 0 \end{pmatrix} \equiv \begin{pmatrix} a & cx \\ b & 0 \end{pmatrix} \begin{pmatrix} dx & fx \\ e & 0 \end{pmatrix}. \tag{66}$$

Other useful matrix transformations include

$$\begin{pmatrix} a & cx \\ b & 0 \end{pmatrix} \begin{pmatrix} d & f \\ e & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} \begin{pmatrix} d & f \\ ex & 0 \end{pmatrix} \tag{67}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv \begin{pmatrix} ax & cx \\ bx & dx \end{pmatrix} \tag{68}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} \equiv \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} f & h \\ e & g \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} ax & c \\ bx & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} ex & gx \\ f & h \end{pmatrix} \tag{70}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix} = \prod_{n=1}^{\infty} \begin{pmatrix} d_n & b_n \\ c_n & a_n \end{pmatrix} \tag{71}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv \left(\begin{array}{c|cc} c & a & c \\ & b & d \\ \hline d & \multicolumn{2}{c}{0} \end{array}\right) \begin{pmatrix} b & d \\ 1 & 0 \end{pmatrix} \tag{72}$$

5.3 Stieltjes type Continued Fraction

The *corresponding* [9] or *Stieltjes type* [13] continued fraction to the power series (46) is given by

$$f(x) = a_0 + \cfrac{a_1 x}{1 + \cfrac{b_0 x}{1 + \cfrac{b_1 x}{1 + \cfrac{b_2 x}{1 + \cfrac{b_3 x}{1 + \ddots}}}}} \tag{73}$$

where

$$b_{2n} = -\frac{C(n+1/n+1)C(n/n-1)}{C(n/n)C(n+1/n)} \tag{74}$$

$$b_{2n+1} = -\frac{C(n+1/n)C(n-1/n-1)}{C(n/n-1)C(n/n)} \tag{75}$$

The sequence of approximants $\left\langle \frac{P_n}{Q_n} \right\rangle_{n=0}^{\infty}$ of the continued fraction (73) occupies the stair step sequence $[0/0], [1/0], [1/1], [2/1], [2/2], \ldots$ in the Padé table for $f(x)$ [see Fig. (2)].

[0/0]	[0/1]	[0/2]	[0/3]	[0/4]
[1/0]	[1/1]	[1/2]	[1/3]	[1/4]
[2/0]	[2/1]	[2/2]	[2/3]	[2/4]
[3/0]	[3/1]	[3/2]	[3/3]	[3/4]
[4/0]	[4/1]	[4/2]	[4/3]	[4/4]

Figure 2: The locations in the Padé table of the convergents of a Stieltjes type continued fraction (73).

[0/0]	[0/1]	[0/2]	[0/3]	[0/4]
[1/0]	[1/1]	[1/2]	[1/3]	[1/4]
[2/0]	[2/1]	[2/2]	[2/3]	[2/4]
[3/0]	[3/1]	[3/2]	[3/3]	[3/4]
[4/0]	[4/1]	[4/2]	[4/3]	[4/4]

Figure 3: The locations in the Padé table of the convergents of a Jacobi type continued fraction (76).

5.4 Jacobi type Continued Fraction

The *associated* [9] or *Jacobi type* [13] continued fraction to the power series (46) is given by

$$f(x) = a_0 + \cfrac{a_1 x}{1 + b_0 x + \cfrac{-b_0 b_1 x}{1 + (b_1 + b_2)x + \cfrac{-b_2 b_3 x}{1 + (b_3 + b_4)x + \cfrac{-b_4 b_5 x}{1 + (b_5 + b_6)x + \ddots}}}} \tag{76}$$

The sequence of approximants $\left\langle \frac{P_n}{Q_n} \right\rangle_{n=0}^{\infty}$ of the continued fraction (76) occupies the diagonal sequence $[0/0], [1/1], [2/2], [3/3], [4/4], \ldots$ in the Padé table for $f(x)$ [see Fig. (3)].

[0/0]	[0/1]	[0/2]	[0/3]	[0/4]
[1/0]	[1/1]	[1/2]	[1/3]	[1/4]
[2/0]	[2/1]	[2/2]	[2/3]	[2/4]
[3/0]	[3/1]	[3/2]	[3/3]	[3/4]
[4/0]	[4/1]	[4/2]	[4/3]	[4/4]

Figure 4: The locations in the Padé table of the convergents of a Euler type continued fraction (77).

5.5 Euler type Continued Fraction

The *equivalent* or *Euler type* continued fraction to the power series (46) is given by

$$f(x) = \cfrac{a_0}{1 + \cfrac{-\frac{a_1 x}{a_0}}{1 + \frac{a_1 x}{a_0} + \cfrac{-\frac{a_2 x}{a_1}}{1 + \frac{a_2 x}{a_1} + \cfrac{-\frac{a_3 x}{a_2}}{1 + \frac{a_3 x}{a_2} + \ddots}}}} \tag{77}$$

The sequence of approximants $\left\langle \frac{P_n}{Q_n} \right\rangle_{n=0}^{\infty}$ of a Jacobi type continued fraction occupies the diagonal sequence $[0/0], [1/1], [2/2], [3/3], [4/4], \ldots$ in the Padé table for $f(x)$ [see Fig. (3)].

The sequence of approximants $\left\langle \frac{P_n}{Q_n} \right\rangle_{n=0}^{\infty}$ of the continued fraction (77) occupies the first column $[0/0], [1/0], [2/0], [3/0], [4/0], \ldots$ in the Padé table for $f(x)$ [see Fig. (4)]; namely the partial sums of the Taylor series.

5.6 Generalised Continued Fraction

We have generally assumed above that nothing vanished in such a way as to create difficulties. To deal with the general case where some of the a_n in the formal power series (46) are zero, the *general Stieltjes type* continued fraction is introduced with "gaps" in it. It is

$$f(x) = a_0 + \cfrac{a_1 x^{\alpha_0}}{1 + \cfrac{b_0 x^{\alpha_1}}{1 + \cfrac{b_1 x^{\alpha_2}}{1 + \cfrac{b_2 x^{\alpha_3}}{1 + \cfrac{b_3 x^{\alpha_4}}{1 + \ddots}}}}} \tag{78}$$

where the α_n are positive integers.

6 The Hypergeometric Function

6.1 The Ordinary Hypergeometric Function

In terms of the often used *Pochhammer symbol*

$$\begin{aligned}(a)_0 &:= 1 \\ (a)_n &:= a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\end{aligned} \tag{79}$$

the *ordinary hypergeometric function* is defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{80}$$

where a, b, c and z are complex numbers.

This series converges for $|z| < 1$ so long as c is not a negative integer or zero.

The Euler identity is

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z). \tag{81}$$

Gauss' summation theorem is

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{82}$$

The hyperbolic differential equation

$$z(1-z)\frac{d^2y}{dz^2} + c - (1+a+b)z\frac{dy}{dz} - aby = 0 \tag{83}$$

is satisfied by the ordinary hypergeometric function

$$y_1 = {}_2F_1(a,b;c;z) \tag{84}$$

together with twenty-three other variants [6]

$$y_2 = z^{1-c}\, {}_2F_1(1+a-c,1+b-c,2-c,z) \tag{85}$$

$$y_3 = (1-z)^{-a-b+c}\, {}_2F_1(-a+c,-b+c,c,z) \tag{86}$$

$$y_4 = (1-z)^{-a-b+c} z^{1-c}\, {}_2F_1(1-a,1-b,2-c,z) \tag{87}$$

$$y_5 = {}_2F_1(a,b,1+a+b-c,1-z) \tag{88}$$

$$y_6 = (1-z)^{-a-b+c}\, {}_2F_1(-a+c,-b+c,1-a-b+c,1-z) \tag{89}$$

$$y_7 = z^{1-c}\, {}_2F_1(1+a-c,1+b-c,1+a+b-c,1-z) \tag{90}$$

$$y_8 = (1-z)^{-a-b+c} z^{1-c}\, {}_2F_1(1-a,1-b,1-a-b+c,1-z) \tag{91}$$

$$y_9 = (-z)^{-a}\, {}_2F_1\left(a,1+a-c,1+a-b,\frac{1}{z}\right) \tag{92}$$

$$y_{10} = (-z)^{-b}\, {}_2F_1\left(b,1+b-c,1-a+b,\frac{1}{z}\right) \tag{93}$$

$$y_{11} = (1-z)^{-a-b+c}\,(-z)^{b-c}\;{}_2F_1\left(1-b,-b+c,1+a-b,\frac{1}{z}\right) \quad (94)$$

$$y_{12} = (1-z)^{-a-b+c}\,(-z)^{a-c}\;{}_2F_1\left(1-a,-a+c,1-a+b,\frac{1}{z}\right) \quad (95)$$

$$y_{13} = (1-z)^{-a}\;{}_2F_1\left(a,-b+c,1+a-b,\frac{1}{1-z}\right) \quad (96)$$

$$y_{14} = (1-z)^{-b}\;{}_2F_1\left(b,-a+c,1-a+b,\frac{1}{1-z}\right) \quad (97)$$

$$y_{15} = (1-z)^{-1-a+c}\,(-z)^{1-c}\;{}_2F_1\left(1+a-c,1-b,1+a-b,\frac{1}{1-z}\right) \quad (98)$$

$$y_{16} = (1-z)^{-1-b+c}\,(-z)^{1-c}\;{}_2F_1\left(1+b-c,1-a,1-a+b,\frac{1}{1-z}\right) \quad (99)$$

$$y_{17} = z^{-a}\;{}_2F_1\left(a,1+a-c,1+a+b-c,1-\frac{1}{z}\right) \quad (100)$$

$$y_{18} = z^{-b}\;{}_2F_1\left(b,1+b-c,1+a+b-c,1-\frac{1}{z}\right) \quad (101)$$

$$y_{19} = (1-z)^{-a-b+c}\,z^{b-c}\;{}_2F_1\left(1-b,-b+c,1-a-b+c,1-\frac{1}{z}\right) \quad (102)$$

$$y_{20} = (1-z)^{-a-b+c}\,z^{a-c}\;{}_2F_1\left(1-a,-a+c,1-a-b+c,1-\frac{1}{z}\right) \quad (103)$$

$$y_{21} = (1-z)^{-a}\;{}_2F_1\left(a,-b+c,c,\frac{z}{z-1}\right) \quad (104)$$

$$y_{22} = (1-z)^{-b}\;{}_2F_1\left(b,-a+c,c,\frac{z}{z-1}\right) \quad (105)$$

$$y_{23} = (1-z)^{-1-a+c}\,z^{1-c}\;{}_2F_1\left(1+a-c,1-b,2-c,\frac{z}{z-1}\right) \quad (106)$$

$$y_{24} = (1-z)^{-1-b+c}\,z^{1-c}\;{}_2F_1\left(1+b-c,1-a,2-c,\frac{z}{z-1}\right) \quad (107)$$

whose regions of convergence [see Fig. (5)] cover the entire complex plane. These expressions lead to an analytic continuation of the hypergeometric function to the remainder of the complex plane outside the unit circle with center at the origin in which its definition (80) is valid.

Kummer's twenty-four solutions can be derived [11] from the two fundamental solutions $y_1(z)$ and $y_2(z)$ by considering Euler's identity (81) and the 1-dimensional lfts

$$t_1(z) = z \quad (108)$$

$$t_2(z) = 1-z \quad (109)$$

$$t_3(z) = \frac{1}{z} \quad (110)$$

$$t_4(z) = \frac{1}{1-z} \quad (111)$$

$$t_5(z) = 1-\frac{1}{z} \quad (112)$$

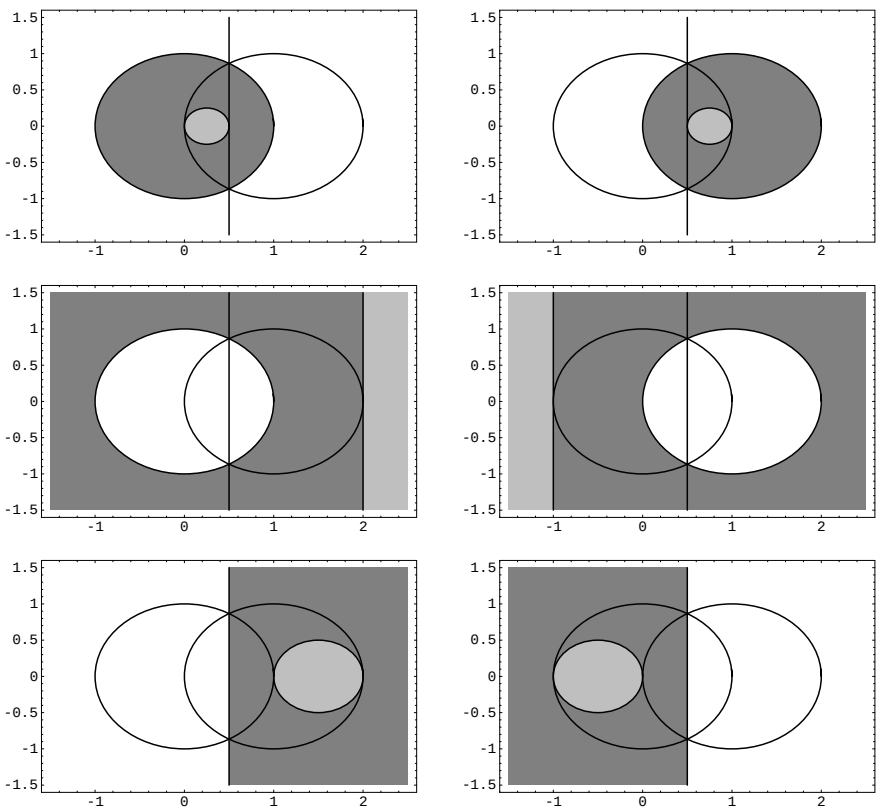

Figure 5: The grey areas show the regions of convergence for Kummer's twenty-four solutions y_{1-4}, y_{5-8}, y_{9-12}, y_{13-16}, y_{17-20} and y_{21-24} of the hypergeometric differential equation respectively. The light grey areas show the resultant mapping of the circle centered at $\frac{1}{4}$ with radius $\frac{1}{4}$ under the transformations $t_1^{-1}(z)$ to $t_6^{-1}(z)$ respectively. Observe that the light grey areas cover the real line.

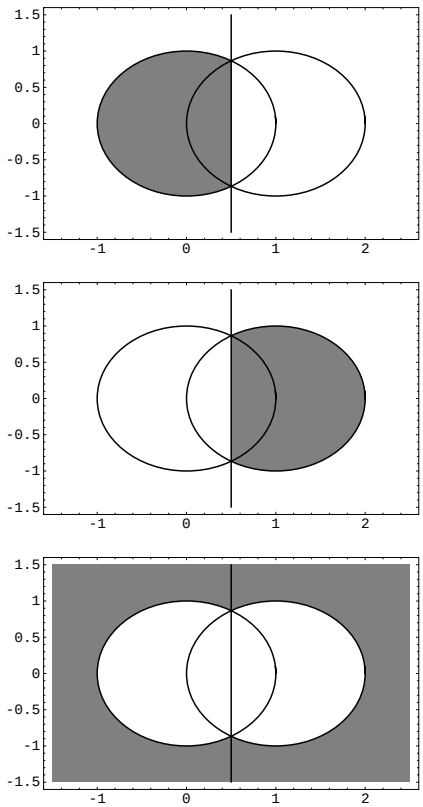

Figure 6: The grey areas show the regions of equality for $y_1 = y_3 = y_{21} = y_{22}$ and $y_2 = y_4 = y_{23} = y_{24}$, for $y_5 = y_7 = y_{17} = y_{18}$ and $y_6 = y_8 = y_{19} = y_{20}$, for $y_9 = y_{11} = y_{13} = y_{15}$ and $y_{10} = y_{12} = y_{14} = y_{16}$ respectively.

$$t_6(z) \quad = \quad \frac{z}{z-1}. \tag{113}$$

These lfts effectively interchange the three regular singular points at $z = 0, 1, \infty$ of the hypergeometric differential equation (83) among themselves.

6.1.1 Analytic continuation formulae

The hypergeometric differential equation can have only two independent solutions in any one domain. Therefore, there must exist linear relationships connecting any three of the twenty-four solutions in their common domain. The coefficients are determined by letting $z \to 0$ and by letting $z \to 1$.

In particular, by Euler's identity (81), we find that the Kummer's twenty-four solutions can be reduced to six equivalent groups, two groups being valid in each of the three regions of convergence [see Fig. (6)].

6.1.2 Trigonometric Identities

$$\log(1+x) \quad = \quad x\,{}_2F_1(1,1;2;-x) \tag{114}$$

$$
\begin{aligned}
(1+x)^a &= {}_2F_1(-a,1;1;-x) && (115)\\
\sin^{-1} x &= x\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) && (116)\\
\sinh^{-1} x &= x\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-x^2\right) && (117)\\
\tan^{-1} x &= x\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-x^2\right) && (118)\\
\tanh^{-1} x &= x\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};x^2\right) && (119)\\
\sin(ax) &= {}_2F_1\left(\frac{1+a}{2},\frac{1-a}{2};\frac{3}{2};\sin^2 x\right) && (120)\\
\cos(ax) &= {}_2F_1\left(\frac{a}{2},-\frac{a}{2};\frac{1}{2};\sin^2 x\right) && (121)\\
\sinh(ax) &= {}_2F_1\left(\frac{1+a}{2},\frac{1-a}{2};\frac{3}{2};\sinh^2 x\right) && (122)\\
\cosh(ax) &= {}_2F_1\left(\frac{a}{2},-\frac{a}{2};\frac{1}{2};\sinh^2 x\right) && (123)\\
& && (124)
\end{aligned}
$$

Theorem 6.1 *The Jacobi type continued fraction to the hypergeometric function* ${}_2F_1(a,b;c;x)$ *with* $-1 < a,b \le 1, c \ge 1$ *and* $x \le 0$ *corresponds to a normal product.*

6.1.3 Nonlinear substitutions of the Hypergeometric Differential Equation

In the hypergeometric differential equation (83), change the variable from z to ω by the substitution

$$\omega = \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1} \text{ or equivalently } z = -\frac{4\omega}{(1-\omega)^2}. \tag{125}$$

This maps the entire z-plane into the interior of the unit circle centered at the origin of the ω-plane. The transformed hypergeometric differential equation function has the solution [12]

$$f(x) = \sum_{n=0}^{\infty} d_n \omega^n \tag{126}$$

where

$$
\begin{aligned}
d_0 &= 1 && (127)\\
d_1 &= \frac{2a(c-2b)}{c} && (128)\\
d_{n+2} &= \frac{2(c-2b)(n+1+a)d_{n+1} + (n+2a)(n+2a+1-c)d_n}{(n+2)(n+1+c)} && (129)
\end{aligned}
$$

which converges for all $|\omega| < 1$. We have

$$ {}_2F_1(a,b;c;z) = (1-\omega)^{2a} f(\omega) \tag{130} $$

This transform may prove interesting because it provides a single formula that converges fairly quickly over the whole complex plane.

6.2 The Confluent Hypergeometric Function

The *confluent hypergeometric function* is defined by

$$ {}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}. \tag{131} $$

This can also be deduced as a special case of the ordinary hypergeometric function. In fact

$$ {}_1F_1(a;c;x) = \lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{x}{b}\right). \tag{132} $$

Some special cases are

$$ \exp x = {}_1F_1(1;1;x) \tag{133} $$

$$ \mathrm{erf} x = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2};\frac{3}{2};-x^2\right). \tag{134} $$

6.3 The Hypergeometric Function

The *hypergeometric function* is defined by

$$ {}_0F_1(;c;x) = \sum_{n=0}^{\infty} \frac{1}{(c)_n} \frac{x^n}{n!}. \tag{135} $$

Again, this can also be deduced as a special case of the confluent hypergeometric function. In fact

$$ {}_0F_1(;c;x) = \lim_{a\to\infty} {}_1F_1\left(a;c;\frac{x}{a}\right). \tag{136} $$

Some special cases are

$$ \sin x = x\, {}_0F_1\left(\frac{3}{2};-\frac{x^2}{4}\right) \tag{137} $$

$$ \sinh x = x\, {}_0F_1\left(\frac{3}{2};\frac{x^2}{4}\right) \tag{138} $$

$$ \cos x = {}_0F_1\left(\frac{1}{2};-\frac{x^2}{4}\right) \tag{139} $$

$$ \cosh x = {}_0F_1\left(\frac{1}{2};\frac{x^2}{4}\right). \tag{140} $$

Theorem 6.2 *The Jacobi type continued fraction to the hypergeometric function* ${}_0F_1(;c;x)$ *with* $c > 0$ *and* $0 \leq x \leq 2(1+c)$ *corresponds to a normal product.*

Theorem 6.3 *The Euler type continued fraction to the hypergeometric function* ${}_0F_1(;c;x)$ *with* $c > 0$ *and* $-c \leq x \leq 0$ *corresponds to a normal product.*

7 The Basic Arithmetic Operations

The spirit of Gosper's [5] Quadratic Algorithm for continued fractions can be extended to normal products. Consider a general 2-dimensional lft

$$t\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}(x,y) = \frac{axy + cx + ey + g}{bxy + dx + fy + h} \tag{141}$$

Clearly, we can define all four basic arithmetic operations by

$$t\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}(x,y) = x + y \tag{142}$$

$$t\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}(x,y) = x - y \tag{143}$$

$$t\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(x,y) = x \times y \tag{144}$$

$$t\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}(x,y) = x \div y \tag{145}$$

Definition 7.1 *Let the* lower part $L : \mathbb{T} \to \mathbb{V}$ *of a tensor* A *be a vector given by*

$$L(A) = any \begin{pmatrix} A_{i_1 \cdots i_N 0} \\ A_{i_1 \cdots i_N 1} \end{pmatrix} \; such\ that \begin{vmatrix} A_{i_1 \cdots i_N 0} & A_{j_1 \cdots j_N 0} \\ A_{i_1 \cdots i_N 1} & A_{j_1 \cdots j_N 1} \end{vmatrix} \leq 0 \tag{146}$$

for all $j_1 \cdots j_N$.

Definition 7.2 *Let the* upper part $U : \mathbb{T} \to \mathbb{V}$ *of a tensor* A *be a vector given by*

$$U(A) = any \begin{pmatrix} A_{i_1 \cdots i_N 0} \\ A_{i_1 \cdots i_N 1} \end{pmatrix} \; such\ that \begin{vmatrix} A_{i_1 \cdots i_N 0} & A_{j_1 \cdots j_N 0} \\ A_{i_1 \cdots i_N 1} & A_{j_1 \cdots j_N 1} \end{vmatrix} \geq 0 \tag{147}$$

for all $j_1 \cdots j_N$.

Definition 7.3 *Let the* range $R : \mathbb{T} \to \mathbb{M}$ *of a tensor to a matrix be given by*

$$R(A) = \begin{pmatrix} L(A)_0 & U(A)_0 \\ L(A)_1 & U(A)_1 \end{pmatrix}. \tag{148}$$

Definition 7.4 *Let the* floor $F : \mathbb{T} \to \mathbb{T}$ *of a tensor be given by*

$$F(A) = \frac{A}{GCD(A)} \tag{149}$$

where $GCD(A)$ *is the greatest divisor of the integers* $A_{i_0 i_1 \cdots i_N}$.

Definition 7.5 *Let the* reverse $\star : \mathbb{M} \to \mathbb{M}$ *of a matrix be given by*

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{\star} = sgn\left(\begin{vmatrix} a & c \\ b & d \end{vmatrix}\right) \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{150}$$

$$sgn(x) = \begin{cases} -1 & if\ x < 0 \\ 0 & if\ x = 0 \\ 1 & if\ x > 0 \end{cases} \tag{151}$$

Remark 7.6 *Some useful identities include*

$$t_A = t_{F(A)} \tag{152}$$

$$A \bullet A^{\star} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{153}$$

$$t_A([0,\infty]) = P(A) \tag{154}$$

$$t_A([0,\infty],[0,\infty]) = P(R(A)) \tag{155}$$

7.1 Operational Semantics

The BNF specification of ρ-notation is:

$$\begin{aligned} < \rho\text{-term} > \ ::= \ & < \text{vector} > | \\ & < \text{matrix} > [< \rho\text{-term} >]| \\ & < \text{3-tensor} > [< \rho\text{-term} >, < \rho\text{-term} >] \end{aligned} \tag{156}$$

The class of ρ-terms consists of words constructed from the following alphabet:

$\mathbb{V}$	vectors
$\mathbb{M}$	matrices
$\mathbb{T}^3$	rank 3 tensors
[ ]	*parentheses*
,	*comma*

Definition 7.7 *The class* Σ *of* ρ*-terms is the least class satisfying the following:*

(i) *if* $A \in \mathbb{V}$ *then* $A \in \Sigma$

(ii) *if* $A \in \mathbb{M}$ *and* $x \in \Sigma$ *then* $A[x] \in \Sigma$

(iii) *if* $A \in \mathbb{T}^3$ *and* $x, y \in \Sigma$ *then* $A[x,y] \in \Sigma$

Simplification Rule

$$A \to F(A) \tag{157}$$

Absorption Rules

$$
\begin{aligned}
A[B[x]] &\rightarrow (A \bullet B)[x] && (158)\\
A[B] &\rightarrow A \bullet B && (159)\\
A[B[x], y] &\rightarrow (A \bullet_1 B)[x, y] && (160)\\
A[x, B[y]] &\rightarrow (A \bullet_2 B)[x, y] && (161)\\
A[B, y] &\rightarrow (A \bullet_1 B)[y] && (162)\\
A[x, B] &\rightarrow (A \bullet_2 B)[x] && (163)
\end{aligned}
$$

Emission Rules

$$
\begin{aligned}
A[x, y] &\rightarrow R(A)[((R(A)^\star \bullet A)[x, y]] && (164)\\
\text{Pif } A < B : C[x] : D[y] &\rightarrow \begin{cases} C[x] & \text{if } A < B \\ D[y] & \text{if } A > B \\ E & \text{if } A \simeq B \end{cases} . && (165)
\end{aligned}
$$

where $E = F[\text{Pif } A < B : (F^\star \bullet C)[y] : (F^\star \bullet D)[z]]$ and $F = C \vee D$.

Fixed Point Combinator

$$Y : F \rightarrow F : (Y : F) \qquad (166)$$

Iteration Combinators

$$
\begin{aligned}
\Psi_0^1 : F : A : n &\rightarrow A_n[F : A : n + 1] && (167)\\
\Psi_0^2 : F : A : B : n &\rightarrow A_n[B_n[F : A : B : n + 1]] && (168)\\
\Psi_1^1 : F : A : n : x &\rightarrow A_n[x, F : A : n + 1 : x] && (169)\\
\Psi_1^2 : F : A : B : n : x &\rightarrow A_n[x, B_n[x, F : A : B : n + 1 : x]] && (170)\\
\Psi_2^2 : F : A : B : n : x : y &\rightarrow A_n[x, B_n[y, F : A : B : n + 1 : x : y]]. && (171)
\end{aligned}
$$

where A_n and B_n are tensor functions of n.

Note that

$$
\begin{aligned}
Y : \Psi_1^1 : A : 1 : x &\rightarrow \Psi_1^1 : (Y : \Psi_1^1) : A : 1 : x\\
&\rightarrow A_1[x, Y : \Psi_1^1 : A : 2 : x]\\
&\vdots\\
&\rightarrow A_1[x, A_2[x, A_3 \ldots]]].
\end{aligned}
$$

These rules can easily be extended to higher dimensions.

8 Quantisation Redundancy

In order to control the size of the integers at the head of a normal product compared to the information held, we need to round to an appropriately in order to represent intervals with B-iadic end points.

Definition 8.1 *The* Quantisation function $Q_q : \mathbb{M} \to \mathbb{M}$ *is defined by*

$$Q_q\left(\left(\begin{array}{cc} a & c \\ b & d \end{array}\right)\right) = \left(\begin{array}{cc} l & u \\ q & q \end{array}\right) \tag{172}$$

where

$$l = \sup\{p : \left|\begin{array}{cc} p & a \\ q & c \end{array}\right| \leq 0\} \tag{173}$$

$$u = \inf\{p : \left|\begin{array}{cc} b & p \\ d & q \end{array}\right| \leq 0\} \tag{174}$$

and we call the difference between u and l the quantisation redundancy *at level q, denoted $R_q : \mathbb{M} \to \mathbb{Z}$.*

Clearly, the quantisation function is monotone.

Definition 8.2 *The* B-adic Level n Redundant Quantisation function ${}_B\mathcal{Q}^n : \mathbb{M} \to \mathbb{M}$ *is defined by*

$${}_B\mathcal{Q}^n(A) = Q_{\sup\{B^m : R_{B^m} \leq B^n\}}(A) \tag{175}$$

Note that we have the fairly simple equivalence

$$\left(\begin{array}{cc} a & b \\ B^p & B^p \end{array}\right) \bullet \left(\begin{array}{cc} (b-a)B^q - c & d \\ c & (b-a)B^q - d \end{array}\right) \equiv \left(\begin{array}{cc} aB^q + c & bB^q - d \\ B^{p+q} & B^{p+q} \end{array}\right) \tag{176}$$

and so quantisation should not make the incremental representation more complicated.

Some identities that may be useful to simplify the representation are

$$t\left(\begin{array}{cc} bn+a & dn+c \\ b & d \end{array}\right)(x) = n + t\left(\begin{array}{cc} a & c \\ b & d \end{array}\right)(x) \tag{177}$$

$$t\left(\begin{array}{cc} pa & pc \\ qb & qd \end{array}\right)(x) = \frac{p}{q} \times t\left(\begin{array}{cc} a & c \\ b & d \end{array}\right)(x) \tag{178}$$

$$t\left(\begin{array}{cc} bp+aq & dp+cq \\ bq & dq \end{array}\right) = \frac{p}{q} + t\left(\begin{array}{cc} a & c \\ b & d \end{array}\right)(x). \tag{179}$$

9 The Elementary Functions

Observe that a 1-dimensional lft with linear coefficients is a 2-dimensional lft.

$$\begin{aligned} {}^t\begin{pmatrix} ay+c & ey+g \\ by+d & fy+h \end{pmatrix}(x) &= \frac{(ay+c)x+(by+d)}{(ey+g)x+(fy+h)} \\ &= {}^t\begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}(x,y) \end{aligned}$$

This means that we can extend the algorithm in the last section to include the evaluation of normal products with linear coefficients in other normal products.

9.1 Rational Function

The rational function $\mathrm{rat}(x,y)$ is given by

$$\mathrm{rat} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{V} \tag{180}$$

$$\mathrm{rat}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{181}$$

9.2 Power Function

The Stieltjes type continued fraction to the power function is

$$(1+x)^y = 1 + \cfrac{xy}{1 + \cfrac{\frac{x(1-y)}{2}}{1 + \cfrac{\frac{x(1+y)}{6}}{1 + \cfrac{\frac{x(2-y)}{6}}{1 + \cfrac{\frac{x(2+y)}{10}}{1 + \cfrac{}{\ddots}}}}}}. \tag{182}$$

The equivalent normal product is

$$(1+x)^y = \begin{pmatrix} 1 & 1 \\ xy & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 1 & \frac{x(n-y)}{4n-2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{x(n+y)}{4n+2} \\ 1 & 0 \end{pmatrix} \tag{183}$$

where $x \geq 0$ and $0 \leq y \leq 1$.

This can be transformed to

$$(1+x)^y = \begin{pmatrix} y & 1 \\ 0 & 1 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ x & 2n-1 \end{pmatrix} \begin{pmatrix} 0 & n-y \\ n+y & 2 \end{pmatrix} \tag{184}$$

where $x \geq 0$ and $0 \leq y \leq 1$.

From Kummer's twenty-four solutions (84) to (107) we can derive the trivial identity

$$(1+x)^y = \left(1 - \frac{x}{1+x}\right)^{-a}. \tag{185}$$

This allows us to evaluate the power function efficiently by

$$\mathrm{pow}[x,y] \to \begin{cases} \perp & \text{if } x = 0 \text{ and } y = 0 \\ 0 & \text{if } x = 0 \text{ and } y \neq 0 \\ 1 & \text{if } x \neq 0 \text{ and } y = 0 \\ \mathrm{rpow}[x,y] & \text{if } 0 < |x| < 1 \\ \mathrm{rpow}\left[\frac{1}{x}, -y\right] & \text{if } |x| > 1 \end{cases} \tag{186}$$

where

$$\begin{aligned} \mathrm{rpow}[x,y] \quad \to \quad & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \Bigg[y, Y : \Psi_2^2 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2n-1 \end{pmatrix} : \\ & \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & n & n & 2 \end{pmatrix} : 1 : x - 1 : y \Bigg]. \end{aligned}$$

9.3 Trigonometric Functions

The continued fraction expansion

$$\tan(k\phi) = \cfrac{k\tan(\phi)}{1 + \cfrac{(1-k^2)\tan^2(\phi)}{3 + \cfrac{(4-k^2)\tan^2(\phi)}{5 + \cfrac{(9-k^2)\tan^2(\phi)}{7 + \cfrac{(16-k^2)\tan^2(\phi)}{9+ \ddots}}}}} \tag{187}$$

by Euler [1, 2] has the equivalent normal product with $\phi = \frac{\pi}{4}$

$$\tan\left(\frac{\pi}{4}x\right) = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 2n-1 & n^2 - x^2 \\ 1 & 0 \end{pmatrix} \tag{188}$$

where $0 \leq x \leq 1$.

This can be simplified to

$$\tan\left(\frac{\pi}{4}x\right) = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & n-1+x \\ n-x & 2n-1 \end{pmatrix} \tag{189}$$

where $0 \leq x \leq 1$.

$$\tan x = x + \cfrac{5x^3}{15 - 7x^2 + \cfrac{x^2}{1 + \cfrac{9x^2}{315 - 28x^2 + \cfrac{5x^2}{1 + \cfrac{13x^2}{1287 - 44x^2 + \ddots}}}}} \tag{190}$$

by Schlömilch [7] corresponds to the normal product

$$\begin{aligned}
\tan x \quad = \quad & \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5x^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 15 - 7x^2 & x^2 \\ 1 & 0 \end{pmatrix} \\
& \prod_{n=2}^{\infty} \begin{pmatrix} 1 & (4n+1)x^2 \\ 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} (4n-3)(4n-1)(4n+1) - 4(4n-1)x^2 & (4n-3)x^2 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

where $x^2 \leq \frac{15}{7}$.

Therefore, the operational semantics is

$$\begin{aligned}
\tan[x] \quad \rightarrow \quad & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left[x, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{pmatrix} \right. \\
& \left[x^2, \begin{pmatrix} -7 & 15 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \left[x^2, Y : \Psi_1^2 : \begin{pmatrix} 0 & 1 & 0 & 1 \\ 4n+1 & 0 & 0 & 0 \end{pmatrix} : \right. \right. \\
& \left. \left. \left. \begin{pmatrix} -4(4n-1) & (4n-3)(4n-1)(4n+1) & 0 & 1 \\ 4n-3 & 0 & 0 & 0 \end{pmatrix} : 2 : x^2 \right] \right] \right] .
\end{aligned}$$

Sine and cosine can be evaluated using the identities

$$\begin{aligned}
\sin x &= \frac{\tan x}{\sqrt{1 + \tan x^2}} && (191) \\
&= \frac{2t}{1 + t^2} && (192) \\
\cos x &= \frac{1}{\sqrt{1 + \tan x^2}} && (193) \\
&= \frac{1 - t^2}{1 + t^2} && (194) \\
t &= \tan\left(\frac{x}{2}\right) . && (195)
\end{aligned}$$

The generalised Jacobi type continued fraction to the sine function is

$$\sin x = \cfrac{x}{1+\cfrac{\frac{x^2}{6}}{1-\cfrac{\frac{31x^2}{294}}{1+\cfrac{\frac{\frac{7x^2}{360}}{1-\cfrac{\frac{551x^2}{1728720}}{1+\cfrac{\frac{\frac{x^2}{9072}}{1-\cfrac{\frac{14561x^2}{593978}}{1+\cfrac{\frac{\frac{3354743x^2}{68769269712}}{1-\frac{2705x^2}{311454}+\ddots}}{1-\frac{17x^2}{990}}}}}}}}}}. \tag{196}$$

The generalised Jacobi type continued fraction to the cosine function is

$$\cos x = \cfrac{x}{1+\cfrac{\frac{x^2}{2}}{1-\cfrac{\frac{61x^2}{150}}{1+\cfrac{\frac{\frac{5x^2}{24}}{1-\cfrac{\frac{5x^2}{126}}{1+\cfrac{\frac{\frac{313x^2}{630000}}{1-\cfrac{\frac{719x^2}{15650}}{1+\cfrac{\frac{\frac{59x^2}{317520}}{1-\frac{1165x^2}{81774}+\ddots}}{1+\frac{73075x^2}{1086280272}}}}}}}}}}. \tag{197}$$

9.4 Inverse Trigonometric Functions

The general Stieltjes type continued fraction to the inverse tangent function is

$$\tan^{-1} x = \cfrac{x}{1+\cfrac{\frac{x^2}{3}}{1+\cfrac{\frac{4x^2}{15}}{1+\cfrac{\frac{9x^2}{35}}{1+\cfrac{\frac{16x^2}{63}}{1+\ddots}}}}}. \tag{198}$$

The equivalent normal product is

$$\tan^{-1} x = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 1 & \frac{n^2 x^2}{4n^2-1} \\ 1 & 0 \end{pmatrix} \tag{199}$$

where $x \geq 0$.

This can be transformed to

$$\tan^{-1} x = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ n^2 x & 2n-1 \end{pmatrix} \tag{200}$$

where $x \geq 0$.

Therefore, the operational semantics is

$$\tan^{-1}[x] \quad \rightarrow \quad Y : \Psi_1^1 : \begin{pmatrix} 0 & 0 & n^2 & 0 \\ 1 & 0 & 0 & 2n-1 \end{pmatrix} : 1 : x.$$

Also, using the identity $\tan^{-1}(1) = \frac{\pi}{4}$, we have the normal product

$$\frac{\pi}{4} = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ n^2 & 2n-1 \end{pmatrix} \tag{201}$$

or more usefully we can use the identity

$$\frac{\pi}{4} = 12 \tan^{-1}\left(\frac{1}{18}\right) + 8 \tan^{-1}\left(\frac{1}{57}\right) - 5 \tan^{-1}\left(\frac{1}{239}\right). \tag{202}$$

From Kummer's twenty-four solutions (84) to (107) we can derive the identities

$$\begin{aligned} \sin^{-1} x &= \frac{\pi}{2} - \sin^{-1}\left(\sqrt{1-x^2}\right) && (203) \\ &= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) && (204) \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) && (205) \end{aligned}$$

$$\begin{aligned} \tan^{-1} x &= \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) && (206) \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right) && (207) \\ &= \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) && (208) \end{aligned}$$

9.5 Exponential Function

The Stieltjes type continued fraction to the exponential function is

$$\exp x = 1 + \cfrac{x}{1 + \cfrac{-\frac{x}{2}}{1 + \cfrac{\frac{x}{6}}{1 + \cfrac{-\frac{x}{6}}{1 + \cfrac{\frac{x}{10}}{1+ \ddots}}}}}. \tag{209}$$

Therefore, the Jacobi type continued fraction to the exponential function [1, 2] is

$$\exp x = 1 + \cfrac{x}{1 - \frac{x}{2} + \cfrac{\frac{x}{12}}{1 + \cfrac{\frac{x}{60}}{1 + \cfrac{\frac{x}{140}}{1 + \cfrac{\frac{x}{252}}{1+ \ddots}}}}}. \tag{210}$$

The equivalent normal product is

$$\exp x = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{x}{2} & \frac{x}{12} \\ 1 & 0 \end{pmatrix} \prod_{n=2}^{\infty} \begin{pmatrix} 1 & \frac{x}{(4n+2)(4n-2)} \\ 1 & 0 \end{pmatrix} \tag{211}$$

where $x \geq 0$.

This normal product can be linearised to

$$\exp x = \begin{pmatrix} x & 2+x \\ x & 2-x \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ x & 4n+2 \end{pmatrix} \tag{212}$$

where $0 \leq x \leq 2$.

Therefore, the operational semantics is

$$\exp[x] \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix} \left[x, Y : \Psi_1^1 : \begin{pmatrix} 0 & 4n+2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : 1 : x \right].$$

Here is an incredibly simple equation for the natural number

$$e - 1 = \prod_{n=2}^{\infty} \begin{pmatrix} n & 1 \\ 1 & 1 \end{pmatrix}. \tag{213}$$

9.6 Logarithmic Function

The Stieltjes type continued fraction to the logarithmic function is

$$\log(1+x) = \cfrac{x}{1+\cfrac{\frac{x}{2}}{1+\cfrac{\frac{x}{6}}{1+\cfrac{\frac{x}{3}}{1+\cfrac{\frac{x}{5}}{1+\ddots}}}}}. \tag{214}$$

The equivalent normal product is

$$\log(1+x) = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{2} \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 1 & \frac{nx}{4n+2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{(n+1)x}{2n+1} \\ 1 & 0 \end{pmatrix} \tag{215}$$

where $x \geq 0$.

This can be transformed to

$$\log(1+x) = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ x & 2n-1 \end{pmatrix} \begin{pmatrix} 0 & n \\ n & 2 \end{pmatrix} \tag{216}$$

where $x \geq 0$.

From Kummer's twenty-four solutions (84) to (107) we can derive the trivial identity

$$\log(1+x) = -\log\left(1 - \frac{x}{1+x}\right). \tag{217}$$

This allows us to evaluate the logarithm efficiently by

$$\log[x] \to \begin{cases} \perp & \text{if } x < 0 \\ -\infty & \text{if } x = 0 \\ \text{rlog}\left[\frac{1}{x}\right] & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{if } x = 1 \\ \text{rlog}[x] & \text{if } 1 < x < 2 \end{cases} \tag{218}$$

where

$$\text{rlog}[x] \to Y : \Psi_1^1 : \begin{pmatrix} n & 0 & 0 & (2n-1)n \\ 2 & 0 & n & 4n-2 \end{pmatrix} : 1 : x-1. \tag{219}$$

9.7 Hyperbolic Functions

The Stieltjes type continued fraction to the hyperbolic tangent function is

$$\tanh x = \cfrac{x}{1+\cfrac{\frac{x^2}{3}}{1+\cfrac{\frac{x^2}{15}}{1+\cfrac{\frac{x^2}{35}}{1+\cfrac{\frac{x^2}{63}}{1+\ddots}}}}}. \tag{220}$$

The equivalent normal product is

$$\tanh x = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} \prod_{n=1}^{\infty} \begin{pmatrix} 1 & \frac{x^2}{4n^2-1} \\ 1 & 0 \end{pmatrix} \tag{221}$$

where $x \geq 0$.

This can be transformed to

$$\tanh x = \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ x & 2n-1 \end{pmatrix} \tag{222}$$

where $x \geq 0$.

Therefore, the operational semantics is

$$\tanh[x] \quad \rightarrow \quad Y : \Psi_1^1 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2n-1 \end{pmatrix} : 1 : x.$$

Hyperbolic sine and cosine can be evaluated using the identities

$$\sinh x = \frac{\tanh x}{\sqrt{1-\tanh x^2}} \tag{223}$$

$$\cosh x = \frac{1}{\sqrt{1-\tanh x^2}}. \tag{224}$$

The generalised Jacobi type continued fraction to the hyperbolic sine func-

tion is

$$\sinh x = \cfrac{x}{1-\frac{x^2}{6}+\cfrac{\frac{7x^2}{360}}{1+\frac{31x^2}{294}+\cfrac{\frac{551x^2}{1728720}}{1+\frac{17x^2}{990}+\cfrac{\frac{x^2}{9072}}{1+\frac{14561x^2}{593978}+\cfrac{\frac{3354743x^2}{68769269712}}{1+\frac{2705x^2}{311454}+\ddots}}}}}. \tag{225}$$

The generalised Jacobi type continued fraction to the cosine function is

$$\cosh x = \cfrac{x}{1-\frac{x^2}{2}+\cfrac{\frac{5x^2}{24}}{1+\frac{61x^2}{150}+\cfrac{\frac{313x^2}{630000}}{1+\frac{5x^2}{126}+\cfrac{\frac{59x^2}{317520}}{1+\frac{719x^2}{15650}+\cfrac{\frac{73075x^2}{1086280272}}{1+\frac{1165x^2}{81774}+\ddots}}}}}. \tag{226}$$

9.8 Inverse Hyperbolic Functions

Using the identity

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

and (216), we have the normal product

$$\tanh^{-1} x \quad = \quad \prod_{n=1}^{\infty} \begin{pmatrix} 0 & x \\ x & (2n-1)(1-x) \end{pmatrix} \begin{pmatrix} 0 & n \\ n & 1 \end{pmatrix} \tag{227}$$

where $0 \le x \le 1$.

Therefore, the operational semantics is

$$\tanh^{-1}[x] \quad \to \quad Y : \Psi_1^1 : \begin{pmatrix} n & 0 & n(1-2n) & 2n-1 \\ 1 & n & 1-n & 2n-1 \end{pmatrix} 1 : x$$

Note also

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1}) \tag{228}$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2-1}) \tag{229}$$

$$\sqrt{x} = \prod_{n=1}^{\infty} \begin{pmatrix} a & x \\ 1 & a \end{pmatrix} \tag{230}$$

$$\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \tag{231}$$

$$\cos^{-1} x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \tag{232}$$

$$x + \sqrt{x^2+y} = \prod_{n=1}^{\infty} \begin{pmatrix} 2x & y \\ 1 & 0 \end{pmatrix} \tag{233}$$

The generalised Stieltjes type continued fraction to the inverse hyperbolic sine function is

$$\sinh^{-1} x = \cfrac{x}{1 + \cfrac{\frac{x^2}{6}}{1 + \cfrac{\frac{17x^2}{60}}{1 + \cfrac{\frac{549x^2}{2380}}{1 + \cfrac{\frac{69049x^2}{261324}}{1+ \ddots}}}}}. \tag{234}$$

From Kummer's twenty-four solutions (84) to (107) we can derive the identities

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \tag{235}$$

$$\tanh^{-1} x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \tag{236}$$

10 Conclusion and Future Directions

Linear fractional transformations seem to provide a very promising route to the holy grail of a feasible representation of the computable real numbers.

The main areas of future work are:

Preliminary implementation in Mathematica Virtually, all the formulae and rules in this report have been evaluated theoretically and verified empirically using Mathematica.

Intermediate implementation in Haskell Juarez Muylaert-Filho is currently implementing EXACT, which is a subset of Haskell extended with an exact real number data type, based initially on Martin Escardo's work. However, it is being designed in an modular manner, so that the underlying representation of the exact real number data type can be exchanged to something involving linear fractional transformations.

Final implementation in C++ This implementation will be most interesting from a commercial point of view, due to the widespread acceptance of C/C++ as the de facto computer language. It will also serve as an interesting exercise to see how a highly optimised functional language implementation compares to a brute force imperative language implementation.

In any case, we may be able to translate some of the results of abstract interpretation and static analysis by Chris Hankin and Juarez Muylaert-Filho in the functional approach to this imperative one.

Operational and Denotational Semantics Once, our feasibility concerns have been allayed, we hope to retreat back into domain theory, examine various domains and construct a denotational semantics for the established real number representation. Further, we will seek to prove that the operational semantics is sound and complete with respect to this denotational semantics. This is also known as the Adequacy Property.

Complexity Analysis Conduct a complexity analysis on all the basic arithmetic operations together with the elementary functions, so that a formal comparison can be made with floating point. Hopefully, this will be backed up by empirical evidence using the final C++ implementation.

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